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Non-isothermal flow of non-Newtonian fluids through a porous medium

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Abstract—The question of the rheological effects of non-Newtonian fluids on some non-isothermal flows through a porous medium is addressed. These effects are illustrated on the temperature and pressure distributions for the case of a power law fluid with a yield stress, in which the yield stress is temperature dependent. The steady state solutions for the radial flow are analytically obtained. The temperature effect on the pressure distributions is graphically illustrated. The unsteady state solutions for one-dimensional flow, in which non-Newtonian behavior is of power law only and rheological parameters are temperature independent, are also obtained. These solutions determine the pressure and temperature distributions when the basic equations are decoupled and the fluid flow occurs due to the decompression mechanism of a non-Newtonian oil.

1. INTRODUCTION

THE FLOW of non-Newtonian fluids through a porous medium is a topic of special interest in many engineering applications. Recently, with the increasing interest in the production of heavy crude oil by means of thermal methods, as for example the steam injection into oil reservoir, it has become essential to have an adequate understanding of the rheological effects on the flow behavior in a porous medium. From a rheological point of view, these complex fluids are non-Newtonian of power law with a yield stress, in which the apparent viscosity is a monotonic decreasing function of increasing shear rate. As a result, the following rheological equation can be used:

$$\tau = H(\dot{\gamma})^n + \tau_0; \quad |\tau| > \tau_0 \quad \text{for} \quad \dot{\gamma} \neq 0$$
$$\text{and} \quad |\tau| \leqslant \tau_0 \quad \text{for} \quad \dot{\gamma} = 0 \quad (1)$$

from which the apparent viscosity is expressed as

$$\mu_{ap} = H(\dot{\gamma})^{n-1} + \frac{\tau_0}{\dot{\gamma}}; \quad \dot{\gamma} \neq 0 \quad (2)$$

where for a shear thinning fluid $n < 1$. In the above equations τ is the shear stress, τ_0 the yield stress, $\dot{\gamma}$ the shear rate, and H and n are the rheological parameters.

On the other hand, from the capillary tube model of pore space geometry, $\dot{\gamma}$ is expressed in terms of fluid velocity by the relation

$$\dot{\gamma} = \frac{3n+1}{n} \frac{v}{\sqrt{(8k\phi)}} \quad (3)$$

where k is the permeability and ϕ is the porosity. Considering the case of steady plane radial flow of an incompressible fluid, then from the equation of continuity one has

$$v = \frac{Q_0}{2\pi h R} \quad (4)$$

in which Q_0 = constant is the volumetric flow rate and h is the thickness of the flow system. For a shear thinning fluid without any yield stress, i.e. $\tau_0 = 0$ in (2), the previous relations lead to the relation

$$\mu_{ap} = \mu_w \left(\frac{R}{R_w}\right)^{1-n}; \quad n < 1 \quad (5)$$

which shows that the apparent viscosity is an increasing function of radial distance. Consequently, in non-Newtonian flows through a porous medium, the rheological effects are flow rate dependent. Knowledge of the implications of relation (5) in determining the pressure and temperature distributions in a radial flow of a heated non-Newtonian fluid is relevant in oil reservoir engineering. Particularly, this problem is of great practical interest in the production of heavy crude oils by thermal methods, where a non-isothermal flow of a power law fluid with a yield stress is involved.

The rheological measurements reported in the literature show that the rheological parameters n, H and τ_0 are sensitive to temperature variations. For example, the heavy crude oils produced from Alberta and Venezuela oil sands have a very high viscosity at room temperature. The structure of these complex fluids determines a non-Newtonian behavior. While for light crude oils, which are Newtonian fluids, the viscosity is a constant depending on temperature, the viscosity of heavy crude oils depends strongly on shear rate. It is well known that the fluids with a gel structure at zero rate of shear require the use of the threshold pressure gradient concept in the flow description through a porous medium. The flow rate–pressure

NOMENCLATURE

b	coefficient in relation (6)
h	oil reservoir thickness
H	consistency index (rheological parameter in power law equation)
k	permeability
$l(t)$	pressure front location
n	power law exponent
p	pressure distribution
p_e	pressure at the external boundary
p_w	pressure at the well radius
Δp	pressure drop
Q_0	volumetric flow rate
R	radial distance
R_e	external radius
R_w	well radius
t	time
T	temperature distribution
T_e	temperature at the external radius
T_w	temperature at the well radius
v	velocity.

Greek symbols

α	constant in equation (20)
α_0	threshold pressure gradient
$\alpha_0(T_\infty)$	threshold pressure gradient at the reference temperature T_∞
β^*	compressibility coefficient
$\dot{\gamma}$	shear rate
$\delta(t)$	thermal penetration depth
η	similarity variable
μ_{ap}	apparent viscosity
μ_{ef}	effective viscosity
μ_w	viscosity at the well radius
σ	equivalent thermal diffusivity
τ	shear stress
τ_0	yield stress
ϕ	porosity.

Subscripts

ap	apparent
e	external boundary
ef	effective
R	radial direction
w	well radius.

drop relationship was found to be a curve which does not pass through the origin. This means that a pressure gradient in excess of the threshold gradient will be required to initiate the flow in a porous medium, in which case any structure of the fluid will presumably be broken down.

In conditions of non-isothermal flow, the temperature variation has a significant effect on the yield stress, i.e. on the structure of heavy crude oils. When temperature is increased this structure can no longer exist and the heavy crude oils can behave as Newtonian fluids. As reported from the rheological measurements, the most sensitive rheological parameter to the temperature changes is the yield stress value. The useful empiricism for temperature dependence of τ_0 is expressed by the linear relation

$$\tau_0(T) = \tau_0(T_\infty)+\frac{b}{T_\infty}(T-T_\infty) \tag{6}$$

where $\tau_0(T_\infty)$ is the yield stress value expressed at the reference temperature T_∞ in a porous medium. The case of a cooling effect, i.e. $T < T_\infty$, gives $b < 0$ in (6), since $\tau_0(t) > \tau_0(T_\infty)$, whereas the case of a heating effect, i.e. $T > T_\infty$, also has $b < 0$, since $\tau_0(T) < \tau_0(T_\infty)$.

The main objective in this paper is to address the question of implications of the rheological effects on the non-isothermal flows of power law fluids with a yield stress. Specifically, these effects on the pressure and temperature distributions in a flow system of practical interest will be shown.

2. STEADY STATE SOLUTIONS

In this section the rheological effects of a power law fluid with a yield stress on the pressure and temperature distributions in a non-isothermal steady flow through a porous medium are illustrated. For this purpose, we consider a well located centrally in an oil reservoir producing at a constant pressure or flow rate. In this case we have a plane radial flow, in which case a modified Darcy's law, including the rheological effects associated with equation (1), may be written as [1]

$$|v_R|^n = \frac{k}{\mu_{ef}}\left[\left|\frac{\partial p}{\partial R}\right| - \alpha_0(T)\right] \tag{7}$$

where

$$\left|\frac{\partial p}{\partial R}\right| > \alpha_0 \quad \text{for} \quad v_R \neq 0$$

$$\text{and} \quad \left|\frac{\partial p}{\partial R}\right| < \alpha_0 \quad \text{for} \quad v_R = 0.$$

From the capillary tube model of pore space geometry one has

$$\frac{k}{\mu_{ef}} - \frac{1}{2H}\left(\frac{n\phi}{1+3n}\right)^n\left(\frac{8k}{\phi}\right)^{(1+n)/2} \tag{8}$$

while the threshold pressure gradient $\alpha_0(T)$, which is a function of temperature T, is related to the yield stress $\tau_0(T)$ by the relation

$$\alpha_0(T) = \frac{\beta\tau_0(T)}{\sqrt{k}} \tag{9}$$

β being a fitting parameter to be determined experimentally, reflecting the deviation of a real porous medium from the capillary tube model of pore space geometry, used in deriving relation (9).

The experimental evidence to support the validity of the basic equation (7) has been shown in ref. [2]. For more details on this matter we refer the interested reader to ref. [2].

As shown in the previous section, the yield stress variation with temperature is expressed by relation (6). As a result, instead of (9) we have for $\alpha_0(T)$ in (7), the following relation:

$$\alpha_0(T) = \alpha_0(T_\infty) + \frac{b\beta}{\sqrt{kT_\infty}}(T - T_\infty) \tag{10}$$

so that the modified Darcy's law (7) may now be written as follows:

$$|v_R|^n = \frac{k}{\mu_{ef}}\left[\left|\frac{\partial p}{\partial R}\right| - \frac{b\beta}{\sqrt{kT_\infty}}(T-T_\infty) - \alpha_0(T_\infty)\right] \tag{11}$$

where $\alpha_0(T_\infty)$ is the threshold pressure gradient expressed at a reference temperature T_∞.

Here we are concerned with the case of fluid production, so that in this situation we have $|\partial p/\partial R| > 0$ in (11).

The equation of continuity for an incompressible fluid is

$$\frac{\partial v}{\partial R} + \frac{v}{R} = 0. \tag{12}$$

From (11) and (12) we have

$$\frac{\partial^2 p}{\partial R^2} + \frac{n}{R}\frac{\partial p}{\partial R} - \frac{n\alpha_0(T_\infty)}{R} - \frac{b\beta}{\sqrt{kT_\infty}}\left[\frac{\partial T}{\partial R} + \frac{n}{R}(T-T_\infty)\right] = 0 \tag{13}$$

which by means of the function

$$\pi = p - \alpha_0 R \tag{14}$$

may be rewritten as

$$\frac{1}{R^n}\frac{\partial}{\partial R}\left(R^n\frac{\partial \pi}{\partial R}\right) - \frac{b\beta}{\sqrt{kT_\infty}}\frac{1}{R^n}\frac{\partial}{\partial R}(R^n(T-T_\infty)) = 0. \tag{15}$$

Integration of equations (14) and (15) leads to the relation

$$p(R) = C_1\frac{R^{1-n}}{1-n} + \alpha_0(T_\infty)R + C_2 + \frac{b\beta}{\sqrt{kT_\infty}}\int_{R_w}^{R}(T-T_\infty)\,dR \tag{16}$$

where R_w is the well radius.

As previously pointed out, the apparent viscosity of a power law fluid with a yield stress depends strongly both on the shear rate and on temperature. Equation (16), including the temperature effect on the yield stress, reflects this fact. As a result, the pressure distribution cannot be obtained from (16) unless the temperature distribution is known. For this purpose, one can use the energy equation, which for a steady state becomes

$$v\frac{\partial T}{\partial R} = \frac{\sigma}{R}\frac{\partial}{\partial R}\left(R\frac{\partial T}{\partial R}\right) \tag{17}$$

where σ is the thermal diffusivity and v is the fluid velocity in the porous medium, which in a steady flow is

$$v = \frac{Q_0}{2\pi h R}; \quad Q_0 = \text{constant} \tag{18}$$

Q_0 being the volumetric flow rate and h the oil reservoir thickness. Taking into account (18), the solution of equation (17) may be written as

$$T(R) = C_3 + C_4 R^\alpha \tag{19}$$

where

$$\alpha = \frac{Q_0}{2\pi h\sigma}. \tag{20}$$

To determine the constants C_1, C_2, C_3 and C_4 in (16) and (19), we have the following boundary conditions:

$$R = R_w; \quad P(R_w) = P_w \quad \text{and} \quad T(R_w) = T_w$$
$$R = R_e; \quad P(R_e) = P_e \quad \text{and} \quad T(R_e) = T_e \tag{21}$$

where R_w is the well radius and R_e is the external boundary radius. From (19) and (21) we have

$$C_3 = T_w - \frac{\Delta T R_w^\alpha}{R_e^\alpha - R_w^\alpha} \tag{22}$$

$$C_4 = \frac{\Delta T}{R_e^\alpha - R_w^\alpha} \tag{23}$$

in which $R_e > R_w$ and

$$\Delta T = T_e - T_w; \quad T_e > T_w. \tag{24}$$

Once the temperature distribution is known from (19), the pressure distribution is determined from (16) and expressed, taking into account the conditions (21) related to pressure, as

$$P(R) = C_1\frac{R^{1-n}}{1-n} + \alpha_0 R + C_2 + \frac{b\beta(C_3 - T_\infty)}{\sqrt{kT_\infty}}(R - R_w) + \frac{b\beta C_4}{\sqrt{kT_\infty}(1+\alpha)}(R^{1+\alpha} - R_w^{1+\alpha}); \quad b < 0. \tag{25}$$

From (7) and (25) the volumetric flow rate at the well can be obtained

$$Q_0 = 2\pi h R_w \left[\frac{k}{\mu_{ef}}\left(\frac{C_1}{R_w^n} + \delta_1 C_4 R_w^{\alpha} + \delta_0\right)\right]^{1/n} \quad (26)$$

in which C_3 and C_4 are given by previous relations (22) and (23), while for C_1 and C_2 we have

$$C_1 = \frac{1-n}{R_e^{1-n} - R_w^{1-n}}\left[\Delta p - (\alpha_0 + \delta_0)(R_e - R_w) - \frac{\delta_1 C_4}{1+\alpha}(R_e^{1+\alpha} - R_w^{1+\alpha})\right] \quad (27)$$

$$C_2 = p_w - \alpha_0 R_w - \frac{R_w^{1-n}}{R_e^{1-n} - R_w^{1-n}}\left[\Delta p - (\alpha_0 + \delta_0) \times (R_e - R_w) - \frac{\delta_1 C_4}{1+\alpha}(R_e^{1+\alpha} - R_w^{1+\alpha})\right] \quad (28)$$

where

$$\delta_0 = \frac{b\beta(C_3 - T_\infty)}{\sqrt{kT_\infty}} \quad \text{and} \quad \delta_1 = \frac{b\beta}{\sqrt{kT_\infty}};$$

$$\Delta p = p_e - p_w. \quad (29)$$

In order to evaluate the temperature effect on the yield stress in a steady flow, the pressure distribution (25) and flow rate (26) should be compared to the situation when this effect is neglected. For example, the pressure distribution and flow rate corresponding to an isothermal flow, i.e. $b = 0$ in (25)–(28), will be

$$p(R) = [\Delta p - \alpha_0(R_e - R_w)]\frac{R^{1-n} - R_w^{1-n}}{R_e^{1-n} - R_w^{1-n}} + \alpha_0(R - R_w) + p_w \quad (30)$$

and

$$Q_0 = 2\pi h\left[\frac{(1-n)k}{\mu_{ef}} \frac{\Delta p - \alpha_0(R_e - R_w)}{R_e^{1-n} - R_w^{1-n}}\right]^{1/n}. \quad (31)$$

In order to show the temperature effect on the pressure distribution in a steady flow, we consider the following illustrative example:

$$R_w = 0,\ R_e = 100\ \text{m},\ h = 10\ \text{m},\ \sigma = 10^{-6}\ \text{m}^2\ \text{s}^{-1},$$

$$Q_0 = 10\ \text{m}^3/24\ \text{h},\ T_\infty = T_w = 20°\text{C},\ T_e = 50°\text{C},$$

$$b = 1\ \text{kg m}^{-2},\ k = 1\ \text{Darcy},\ \beta = 10^{-1}$$

$$\text{and} \quad \Delta p = 40\ \text{atm}.$$

Figures 1 and 2 show the dimensionless pressure $(p-p_w)/(p_e-p_w)$ expressed in terms of dimensionless radius R/R_e for $n = 0.5$ and 0.8, when the temperature effect does not exist. For a comparison, the case when this effect is considered is shown in Fig. 3 for $n = 0.8$, from which it can be seen that the pressure profiles are significantly altered; note that the dimensionless group $\Omega = \alpha_0 R_e/\Delta p$ indicates the effects associated with the threshold pressure gradient at T_∞.

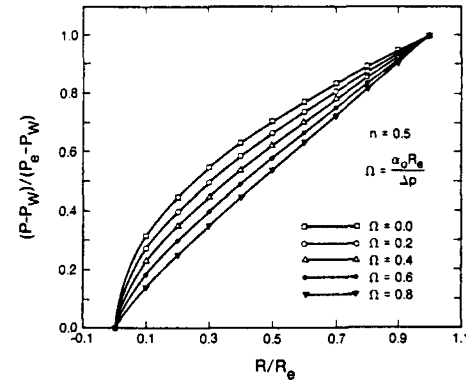

FIG. 1. Effects of threshold pressure gradient on pressure distributions for $n = 0.5$; isothermal flow.

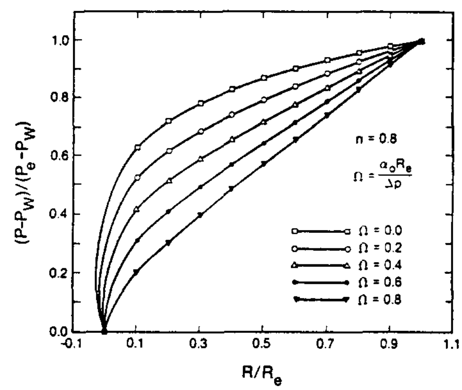

FIG. 2. Effects of threshold pressure gradient on pressure distributions for $n = 0.8$; isothermal flow.

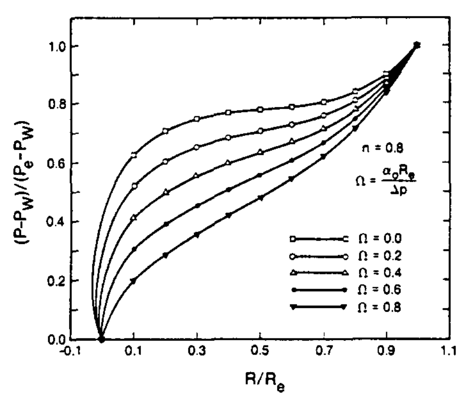

FIG. 3. Effects of threshold pressure gradient on pressure distributions for $n = 0.8$; non-isothermal flow.

3. UNSTEADY STATE SOLUTIONS

This section is concerned with the illustration of the rheological effects on the pressure and temperature distributions in an unsteady flow. A simple case is considered in which the flow is one-dimensional for a power law fluid only, i.e. in the absence of a yield stress effect. As a result, equations (32) and (33), describing the heat transfer and unsteady flow, become decoupled, since the rheological parameters H and n occurring in the effective viscosity μ_{ef} (see relation (8)) are temperature independent. Therefore, the temperature effect on the flow behavior is ignored. Obviously, this case is just a first approximation in an attempt to illustrate analytically the rheological effects of power law fluids on the pressure and temperature distributions. Consequently, for this case the previous equations (7) and (17) become

$$|v|^n = \frac{k}{\mu_{ef}}\frac{\partial p}{\partial x} \tag{32}$$

and

$$\frac{\partial T}{\partial t}+v\frac{\partial T}{\partial x} = \sigma\frac{\partial^2 T}{\partial x^2}. \tag{33}$$

In an unsteady flow, the equation of continuity for a slightly compressible fluid may be written as in ref. [3]

$$\frac{\partial v}{\partial x} = -\beta^*\phi\frac{\partial p}{\partial t} \tag{34}$$

where β^* is the coefficient of fluid compressibility.

Equations (32) and (34) lead to the pressure equation [4]

$$\left(\frac{\partial p}{\partial x}\right)^{(1-n)/n}\frac{\partial^2 p}{\partial x^2} = na^2\frac{\partial p}{\partial t} \tag{35}$$

in which

$$a^2 = \left(\frac{\mu_{ef}}{k}\right)^{1/n}\beta^*\phi \tag{36}$$

with μ_{ef}/k given by relation (8).

As shown in ref. [3], the similarity variable

$$\eta = xt^{-n/(1+n)} \tag{37}$$

reduces the partial differential equation (35) to the ordinary differential equation

$$\frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}+\frac{na^2}{1+n}\eta\left(\frac{\mathrm{d}p}{\mathrm{d}\eta}\right)^{(2n-1)/n} = 0. \tag{38}$$

From (38) we have

$$\frac{\mathrm{d}p}{\mathrm{d}\eta} = \left[C-\frac{na^2}{2}\frac{1-n}{1+n}\eta^2\right]^{n/(1-n)} \tag{39}$$

where C is a constant, while the relations (32), (37) and (39) give the velocity distribution

$$v = \left(\frac{k}{\mu_{ef}}\right)^{1/n} t^{-1/(1+n)}\left[C-\frac{na^2}{2}\frac{1-n}{1+n}\eta^2\right]^{1/(1-n)}. \tag{40}$$

Clearly, this equation shows for $n<1$, i.e. for a shear thinning fluid, the existence of a pressure front at $\eta = \eta_1$ for which one has $v \neq 0$ for $\eta < \eta_1$ and $v = 0$ for $\eta \geqslant \eta_1$. This means $p(x, t) < p_e$ for $0 < x < l(t)$ and $p(x, t) = p_e =$ constant for $x \geqslant l(t)$; $l(t)$ being the front location and p_e the pressure at $t = 0$.

As a result, equation (40) may be rewritten as follows:

$$v = \left(\frac{k}{\mu_{ef}}\right)^{1/n} t^{-1/(1+n)}B(\eta_1^2-\eta^2)^{1/(1-n)}; \quad \eta < \eta_1 \tag{41}$$

in which η_1 is a constant to be determined and B is given by the relation

$$B = \left[\frac{na^2(1-n)}{2(1+n)}\right]^{1/(1-n)}. \tag{42}$$

In terms of front location $l(t)$

$$l(t) = \eta_1 t^{n/(1+n)}; \quad \eta_1 = \text{constant} \tag{43}$$

equation (41) is expressed in the form

$$v = \left(\frac{k}{\mu_{ef}}\right)^{1/n} B\eta_1^{2/(1-n)}t^{-1/(1+n)}\left(1-\frac{x^2}{l^2(t)}\right)^{1/(1-n)} \tag{44}$$

for $0 < x < l(t)$ and $v = 0$ for $x \geqslant l(t)$. At the outface flow, i.e. $x = 0$, equation (44) gives the velocity variation in time for monitoring a constant pressure p_w there

$$v = \left(\frac{k}{\mu_{ef}}\right)^{1/n} B\eta_1^{2/(1+n)}t^{-1/(1+n)}. \tag{45}$$

In order to determine $\eta_1 =$ constant in (45) we can use the condition of pressure continuity at the front location

$$p(l(t), t) = p_e = \text{constant}. \tag{46}$$

Since at the front location we have $v = 0$, or $\mathrm{d}p/\mathrm{d}\eta|_{\eta=\eta_1} = 0$, then

$$C = \frac{na^2}{2}\frac{1-n}{1+n}\eta_1^2$$

in (39). Integration of (39) yields

$$p\left(\frac{\eta}{\eta_1}\right) = p_w + B^*J_n\left(\frac{\eta}{\eta_1}\right); \quad \frac{\eta}{\eta_1} < 1 \tag{47}$$

where

$$J_n\left(\frac{\eta}{\eta_1}\right) = \int_0^{\eta/\eta_1}(1-\xi^2)^{n/(1-n)}\,\mathrm{d}\xi \tag{48}$$

and

$$B^* = \left[\frac{n(1-n)a^2}{2(1+n)}\right]^{n/(1-n)}\eta_1^{(1+n)/(1-n)}. \tag{49}$$

Using (46), i.e. $p(\eta_1) = p_e$, from (47) one obtains

$$\eta_1 = \left(\frac{\Delta p}{J_n(1)}\right)^{(1-n)/(1+n)}\left[\frac{2(1+n)}{(1-n)a^2}\right]^{n/(1+n)} = \text{constant} \tag{50}$$

in which $\Delta p = p_e - p_w$ and

$$J_n(1) = \int_0^1 (1-\xi^2)^{n/(1-n)}\,d\xi = \frac{\sqrt{\pi n}}{1+n}\frac{\Gamma\left(\frac{n}{1-n}\right)}{\Gamma\left(\frac{1+n}{2(1-n)}\right)} \tag{51}$$

Γ being the gamma function.

From previous relations, the pressure distribution (47) may be expressed as

$$\frac{p-p_w}{p_e-p_w} = \frac{J_n\left(\frac{\eta}{\eta_1}\right)}{J_n(1)}. \tag{52}$$

Once η_1 is determined from (50), the pressure front location $l(t)$, given by relation (43), may be known, while the front velocity may also be known from

$$V = \phi\frac{dl}{dt} = \phi\eta_1\frac{n}{1+n}t^{-1/(1+n)}. \tag{53}$$

The most important aspect arising from (53) is that the pressure disturbances in a non-Newtonian fluid with $n<1$, flowing through a porous medium, propagate with a finite velocity. This is in contrast to the infinite velocity of pressure disturbance propagation in a Newtonian fluid, obtained from the parabolic linear equation, i.e. the case $n = 1$ in (35). Consequently, the self-similar solutions (44) and (52) exhibit traveling wave characteristics. From these solutions, it is evident that in a non-Newtonian fluid there exists a moving pressure front. This front separates the disturbed flow domain $0<x<l(t)$ from another $x>l(t)$ which has not felt the effects of pressure disturbances. For further discussion on this matter, the reader is referred to ref. [8]. This relevant result is a consequence of the non-linear effects associated with power law fluids, where the apparent viscosity of a shear thinning fluid is a monotonically increasing function of decreasing velocity. The considerations shown above point out the fundamental differences in the mechanism of pressure disturbance propagation in Newtonian and non-Newtonian fluids flowing through porous media.

According to (53) the front movement is decelerating. To determine the temperature distribution we have the energy equation (33), which integrated over the distance $\delta(t)$ becomes

$$\frac{\partial}{\partial t}\int_0^{\delta(t)} T\,dx - T(\delta(t))\frac{d\delta}{dt} + \int_0^{\delta(t)} v\frac{\partial T}{\partial x}\,dx = \sigma\left[\left.\frac{\partial T}{\partial x}\right|_{\delta(t)} - \left.\frac{\partial T}{\partial x}\right|_0\right] \tag{54}$$

in which $\delta(t)$ is the thermal penetration depth.

The boundary conditions for the temperature distribution in the region $0<x<\delta(t)$ are expressed as

$$x = 0;\quad T(0,t) = T_w = \text{constant}$$
$$x = \delta(t);\quad T(\delta(t),t) = T_f = \text{constant}. \tag{55}$$

A convenient form of the temperature distribution, satisfying the conditions specified in (55), is

$$T(x,t) = T_f - \Delta T\left(1-\frac{x}{\delta(t)}\right)^2;\quad 0<x<\delta(t) \tag{56}$$

where $\Delta T = T_f - T_w$. From (56) one has

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(t)} = 0 \quad\text{and}\quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{2\Delta T}{\delta(t)} \tag{57}$$

and

$$\frac{\partial}{\partial t}\int_0^{\delta(t)} T\,dx = \frac{2T_f+T_w}{3}\frac{d\delta}{dt}. \tag{58}$$

The assumption of a thermal penetration depth $\delta(t)$ gives rise to two new boundary conditions: $T(\delta(t),t) = T_f$ and $\partial T/\partial x|_{x=\delta(t)} = 0$. From (56) and (57) it can be seen that these conditions are satisfied. By substituting relations (44) and (56)–(58) into (54) we obtain

$$\frac{d\delta}{dt} - \frac{6\Omega t^{-1/(1+n)}}{\delta(t)}\int_0^{\delta(t)}\left(1-\frac{x}{\delta(t)}\right)\times\left(1-\frac{x^2}{l^2(t)}\right)^{1/(1-n)}dx = \frac{6\sigma}{\delta(t)} \tag{59}$$

where

$$\Omega = \left(\frac{k}{\mu_{ef}}\right)^{1/n} B\eta_1^{2/(1-n)}. \tag{60}$$

The determination of an analytical expression for $\delta(t)$ from (59) is quite cumbersome, so that in these circumstances it is natural to look for the asymptotic behaviors of $\delta(t)$. For example, taking into account that $x/l(t)<1$, then the following approximate relation:

$$\left(1-\frac{x^2}{l^2(t)}\right)^{1/(1-n)} \cong 1-\frac{1}{1-n}\frac{x^2}{l^2(t)} \tag{61}$$

may be used in (59), which appears to be a reasonable approximation for a long time solution, where $l(t)$ is large. In this case, from (59) we have

$$\delta\frac{d\delta}{dt} + \frac{\Omega t^{-(1+2n)/(1+n)}}{2(1-n)\eta_1^2}\delta^3 - 3\Omega t^{-1/(1+n)}\delta - 6\sigma = 0. \tag{62}$$

Considering the case when $\sigma\to 0$, i.e. the heat transfer mechanism is mainly by convection, then (62) becomes a Riccati equation expressed as

$$\frac{d\delta}{dt} + R(t)\delta^2 = P(t) \tag{63}$$

where

$$R(t) = \frac{\Omega t^{-(1+2n)/(1+n)}}{2(1-n)\eta_1^2} \quad\text{and}\quad P(t) = 3\Omega t^{-1/(1+n)}. \tag{64}$$

The solution of non-linear equation (62) requires a numerical procedure. As observed from numerical computations, the coefficient Ω in (62) is extremely small for the cases of practical interest in oil reservoir engineering. As a result, the terms containing Ω in (62) can be safely neglected. These terms reflect the convection effect in the heat transfer mechanism in a shear thinning fluid flowing through a porous medium. The rheological effects of shear thinning fluid, which are velocity dependent, lead to extremely small velocities in a porous medium, compared to the case of a Newtonian fluid. It should be noted that the apparent viscosity of a shear thinning fluid is an increasing function of decreasing velocity. The numerical results obtained from (62) clearly indicate that the heat transfer mechanism in a non-Newtonian fluid flowing through a porous medium is mainly by conduction. Consequently, the terms in (62) associated with the convection effect may be ignored, in which case we have

$$\delta(t) = (12\sigma t)^{1/2}. \tag{65}$$

On the other hand, if the conduction effect is neglected, i.e. $\sigma = 0$, then, in a first approximation, equation (62) yields

$$\delta(t) \cong \frac{3(1+n)\Omega}{n} t^{n/(1+n)} \tag{66}$$

which is similar to relation (43), giving the location of the pressure front with a different coefficient. It should be kept in mind that previous results are valid for the case when the fluid flow is due to the decompression of a slightly compressible fluid with non-Newtonian behavior.

4. CONCLUDING REMARKS

In this investigation we have analyzed the rheological effects of non-Newtonian fluids in some non-isothermal flows through a porous medium of practical interest in oil reservoir engineering. In the case of a steady flow of a power law fluid with a yield stress, i.e. in the presence of a threshold pressure gradient, the pressure distribution is significantly altered by the yield stress variation with temperature. There are significant differences between the pressure profiles with and without yield stress, as may be seen from Figs. 1–3.

The unsteady state solutions for the case of a shear thinning fluid in the absence of a yield stress, obtained from decoupled equations of heat and fluid flow, indicate that the temperature distribution is not significantly affected by the fluid flow effect, provided that the rheological parameters are temperature independent. It should be pointed out that for a shear thinning fluid, the rheological effects on the viscosity are velocity dependent; the viscosity is a monotonically decreasing function of velocity. The fluid velocity in an oil reservoir containing a non-Newtonian fluid, produced under decompression, will be extremely small. In these circumstances, the heat transfer mechanism in a shear thinning fluid flowing through a porous medium was found to be mainly by conduction. From a fluid mechanics point of view, it is interesting to observe that relation (53), giving the pressure front velocity, is identical to relation (45), giving the fluid velocity at $\eta = 0$, except for the coefficients.

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ECOULEMENT NON ISOTHERME D'UN FLUIDE NON NEWTONIEN A TRAVERS UN MILIEU POREUX

Résumé—On considère les effets rhéologiques d'un fluide non newtonien sur quelques écoulements non isothermes. Ces effets sont illustrés sur les distributions de température et de pression dans le cas d'un fluide à loi puissance pour lequel la tension de cisaillement dépend de la température. Les solutions permanentes pour l'écoulement radial sont obtenues analytiquement. On illustre graphiquement l'effet de la température sur les distributions de pression. Les solutions variables sont obtenues pour un écoulement monodimensionnel dans lequel le comportement non-newtonien est une simple loi puissance avec des paramètres rhéologiques indépendants de la température. Ces solutions déterminent les distributions de pression et de température quand les équations de base sont découplées et l'écoulement du fluide se produit sous l'effet d'un mécanisme de décompression élastique d'une huile non newtonienne.

NICHTISOTHERME STRÖMUNG EINES NICHT-NEWTONSCHEN FLUIDS DURCH EIN PORÖSES MEDIUM

Zusammenfassung—Es wird die Frage der rheologischen Einflüsse eines nicht-newtonschen Fluids auf nichtisotherme Strömungen durch ein poröses Medium erörtert. Diese Einflüsse werden anhand von Temperatur- und Druckverteilungen für den Fall eines "power law"-Fluides mit temperaturabhängiger Fließspannung verdeutlicht. Die stationären Lösungen der radialen Strömungen werden auf analytischem Weg ermittelt. Der Temperatureinfluß auf die Druckverteilung wird grafisch gezeigt. Weiterhin werden instationäre Lösungen für die eindimensionale Strömung ermittelt, bei der sich das nicht-newtonsche Verhalten auf den "power law"-Ansatz beschränkt und die rheologischen Parameter temperaturunabhängig sind. Diese Lösungen bestimmen die Druck- und Temperaturverteilungen, wenn die Grundgleichungen entkoppelt sind und die Strömung aus dem elastischen Dekompressions-Mechanismus eines nicht-newtonschen Öls hervorgeht.

НЕИЗОТЕРМИЧЕСКОЕ ТЕЧЕНИЕ НЕНЬЮТОНОВСКИХ ЖИДКОСТЕЙ ЧЕРЕЗ ПОРИСТУЮ СРЕДУ

Аннотация—Исследуется влияние реологических эффектов на неизотермические течения неньютоновских жидкостей через пористую среду. Приведены распределения температур и давлений в случае степенной жидкости с предельным напряжением сдвига, зависящим от температуры. Аналитически определены стационарные решения для радиального течения. Графически показано влияние температуры на распределения давлений. Получены также нестационарные решения для одномерного течения, при котором поведение неньютоновской жидкости определяется только степенным законом, а реологические параметры не зависят от температуры. Эти решения позволяют найти распределения давлений и температур в случае, когда основные уравнения не взаимосвязаны, а течение жидкости происходит по механизму упругой декомпрессии неньютоновской жидкости типа нефти.